**Barrier Option Pricing Based on Monte-Carlo Simulation**  
**Course**: Asset Pricing  
**Instructor**: Catherine Bruneau

**Author**:

Hassan Abou Brahim

Nayera Ashraf

Vironia Youssef

**Program**: M2 Financial Economics



## Introduction

Barrier options are a type of exotic derivative where the payoff depends not only on the asset price at expiry but also on whether the price crosses a predefined barrier during the option’s life. They are popular due to their lower cost compared to standard options, as the path-dependency reduces the scenarios in which the option will pay off. Barrier options come in two main types:

* Knock-in options: Activated only if the asset price hits the barrier.
* Knock-out options: Expire worthless if the asset price hits the barrier.

These options offer more personalized risk management than traditional options, making them attractive for hedging or speculative purposes, particularly in high-volatility markets such as foreign exchange and commodities. Their lower cost reflects the reduced probability of gain due to the barrier condition.

So, since barrier options are path-dependent, they present a complex pricing challenge. While the Black-Scholes model forms the basis for pricing, the intricacies of barrier options often require numerical methods like Monte Carlo simulations, introduced by Boyle in 1977. These simulations provide the flexibility needed for these complex derivatives.

Monte Carlo methods are widely used to price complex derivatives such as barrier options. Although efficient and flexible, Monte Carlo simulations can suffer from slow convergence, often requiring a large number of simulations to obtain accurate results.

In this project, we apply a simple Monte Carlo simulation approach to the valuation of barrier options and compare the results with analytical formulas where available. The focus is on understanding the basic simulation process without the application of advanced variance reduction techniques, which provides insight into the accuracy and computational requirements of classical Monte Carlo methods.

## Barrier Option Definition Using Formulas

As explained, a barrier option is an exotic derivative whose payoff depends not only on the price of the underlying asset at expiration but also on whether the asset price has crossed a specified barrier level during the life of the option. The general payoff for a barrier option can be described as follows:

***Payoff = 𝑓(𝑆𝑇)×𝐼(Barrier Condition Met)***

Where:

* **𝑆𝑇**  represents the underlying asset price at expiry.
* **𝐼(Barrier Condition Met)** is an indicator function that determines whether the barrier condition has been met at any time t during the life of the option [0,T]. This function equals 1 if the barrier condition is satisfied and 0 otherwise. depends on whether the barrier condition is satisfied
* **𝑓(𝑆𝑇)** represents the payoff function, which could be like that of a standard call or put option. For a **call option**, *f(ST​)=max(ST​−K,0)*, and for a put option,
* *f(ST)=max(K−ST​,0)*.

### Types of Barrier Options

Barrier options are divided into four main types based on whether they are knock-in or knock-out and whether the barrier is above or below the initial price:

**Knock-In and Knock-Out Conditions**

**1. Knock-In Options:** For knock-in options, the barrier must be breached for the option to be activated. The payoff is only valid if the barrier is breached.

**2. Knock-Out Options:** For knock-out options, the option ceases to exist if the barrier is breached. Therefore, the payoff is valid only if the barrier is not breached.

|  |  |  |
| --- | --- | --- |
|  | Option Type |  |
| Barrier Type | In | Out |
| Up | The option becomes active only if the underlying price rises above the barrier H.  ***Payoff=max(ST−K,0)⋅I(St≥H,∃t∈[0,T])*** | The option ceases to exist if the underlying price rises above the barrier H.  ***Payoff=max(ST−K,0)⋅I(St<H,∀t∈[0,T])*** |
| Down | The option becomes active only if the underlying price falls below the barrier H.  ***Payoff=max(K−ST,0)⋅I(St≤H,∃t∈[0,T])*** | The option ceases to exist if the underlying price falls below the barrier H.  ***Payoff=max(K−ST,0)⋅I(St>H,∀t∈[0,T])*** |

1. Price Path Simulation

The Geometric Brownian Motion (GBM) function provided is used to simulate price paths for an asset over time, which follows the stochastic process described by the Black-Scholes model. This process is essential in modeling stock prices and pricing derivatives like barrier options. Here’s a breakdown of the mathematical concepts behind the code and how they relate to the provided equations.

In this context, Monte Carlo simulations generate a large number of potential future paths of the underlying asset's price by simulating random variables that follow the GBM model. Each path represents a possible outcome for the asset price at each point in time until the option's expiration. Monte Carlo methods were first applied to option pricing by Boyle (1977), and since then, they have become essential for handling path-dependent options.

## Mathematical Concepts and Equations

**Time Discretization :** In the simulation of asset prices, time is discretized into small intervals: ***0,T​/n,2T​/n ​,...,(n−1)T/n*​,***T* so where n represents the number of intervals and *T* is the total time horizon (in years). The time step size is denoted by: ***Δt=T/n***

**Stock Price Dynamics (Risk-Neutral Framework):** In the **risk-neutral world**, the asset price dynamics follow a **stochastic differential equation (SDE)**:

***dSt=r\*St\*dt+σ\*St\*dWt***

* ***r*** is the risk-free rate, representing the expected instantaneous return of the stock.
* ***σ*** is the volatility.
* ***dWt***is a Wiener process (standard Brownian motion).

This is the underlying equation of the **Geometric Brownian Motion (GBM)** used to simulate the stock price.

**Logarithmic Price Distribution:** The natural logarithm of the stock price at each time step follows a **normal distribution**:

***uk=ln(Sk/Sk−1)∼N((r-0.5\*σ2)\*T/n,σ2\*T/n)***

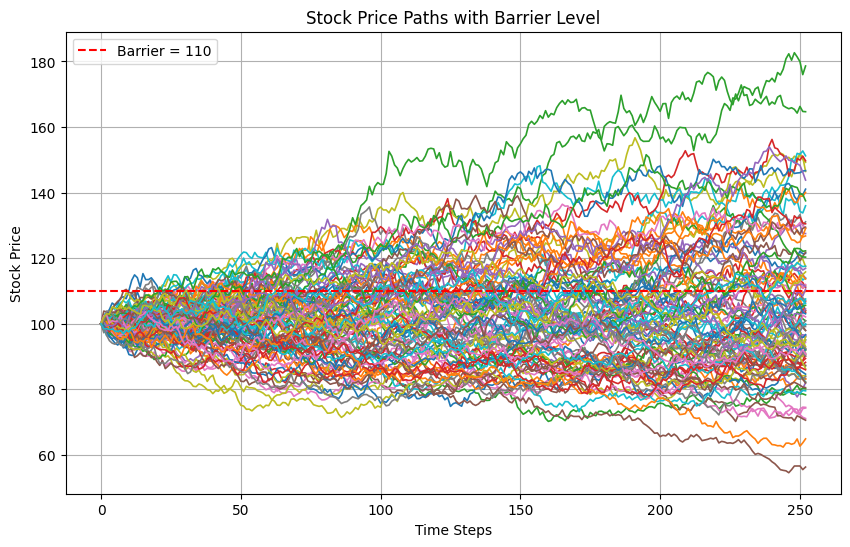
***N*** is a random variable from the standard normal distribution, simulating the randomness in the stock price movement at each step.

**Simulating N Paths:** To generate multiple price paths for Monte Carlo simulation, the code repeats the above process for ***N*** independent paths, each simulating the stock price evolution. In mathematical terms, each simulated path i at time step k is given by:

***Sk(i)=Sk-1(i)\*exp(Sk(i))***

The code repeats this process for each time step ***t=1,2,...,M*** to generate the full path for each simulation. This results in a matrix where each row represents an independent price path.

On the graph below, we plot 100 price paths generated by a Python simulation, based on the mathematical framework outlined above.



**Checking the Barrier Condition:** The key idea of a barrier option is that its payoff depends not only on the underlying asset price at expiry but also on whether the price crosses a certain barrier level during the life of the option. For each simulated path, the code checks whether the price has breached the barrier *H*.

*Down-and-In/Out:* The barrier is breached if the asset price drops below the barrier during its lifetime.

*Up-and-In/Out:* The barrier is breached if the asset price rises above the barrier.

The barrier condition determines whether the payoff is zero (for knock-out options) or if the option activates (for knock-in options). Based on the barrier type, the payoff calculation is adjusted.

|  |  |
| --- | --- |
|  |  |
|  |  |

**Monte Carlo Simulated Price Paths for Barrier Call Options (Up-and-Out, Down-and-In, Up-and-In, and Down-and-Out)**

**Calculating Payoff:** After checking if the barrier is breached, the code calculates the payoff based on whether the option is a call or a put. The payoff depends on the asset price at expiration 𝑆𝑇, the strike price 𝐾, and whether the barrier condition is met.

* For a call option: ***Payoff=max(ST−K,0)***
* For a put option: ***Payoff=max(K−ST,0)***

where *ST* ​ is the terminal asset price at maturity *T*, and *K* is the strike price.

The payoff is then adjusted based on whether the barrier was breached:

* **Knock-Out**: If the barrier is breached, the payoff becomes zero.
* **Knock-In**: If the barrier is not breached, the payoff is set to zero.

**Discount the Payoffs:** Once the payoff is determined, it must be discounted back to the present value. This reflects the time value of money, as the payoff will be received at a future date.The formula for discounting is:

***Discounted Payoff=exp(−r\*T)\*Payoff***

**Option Price and Confidence Interval:** The final price of the option is the average of all discounted payoffs across the N simulations:

***Option Price=(1/N) \*∑Discounted Payoffi***

The Central Limit Theorem is applied to calculate the 95% confidence interval:

***Option Price±1.96\* σOption Price /√N***

The results will display the option price, 95% confidence interval, and the standard deviation of Option Price.

**Convergence of Monte Carlo estimation:** This experiment illustrates how the Monte Carlo estimate for the barrier option price converges as the number of simulations (*N*) increases. At smaller values of 𝑁, the estimates are unstable, reflected in higher standard errors, such as 1.392291 for *N*=101 and 1.118506 for *N*=201. However, as 𝑁increases, the estimates become more stable and the standard error reduces significantly. For instance, by N=1001, the standard error has dropped to 0.446271, and at *N*=5001, it further decreases to 0.213583. The steady decrease in standard error demonstrates the effectiveness of increasing 𝑁 for improving accuracy, following the law of large numbers. Based on these results, an 𝑁 value of around 5000 provides a good balance between accuracy and computational cost, though even N=1000 can offer reasonable precision with a low standard error.

Une image contenant texte, ligne, Tracé, capture d’écran

Description générée automatiquement

## Sensitivity Analysis

Common Assumptions: Under the assumptions 𝑆0=100, *K*=100, *H*=600, *T*=1.0, *r*=0.05, *σ*=0.2, *N*=5000 simulations, and *M*=252-time steps, we explore the sensitivity of the up-and-out call option price by varying the barrier level (*H*), volatility (𝜎), and time to maturity (𝑇).

**Experimenting Barrier Levels:** we observe how varying the barrier level impacts the price of an up-and-out call option. For barrier levels below the strike price (*H* < 100), the option price remains near zero as the likelihood of breaching the barrier early is very high, causing the option to be knocked out and rendered worthless. As the barrier level increases beyond 100, the price of the option rises almost linearly. This reflects the growing probability that the barrier will not be breached, allowing the option to remain active and potentially profitable. Beyond a barrier level of 200, the price stabilizes as the probability of breaching the barrier decreases significantly. Any further increase in the barrier level has a negligible effect on the price. This result indicates that barrier options with low barriers are nearly worthless, whereas higher barriers behave similarly to a standard European option.

Une image contenant texte, ligne, Tracé, diagramme

Description générée automatiquement

**Exploring the Effect of Volatility on Option Price:** When analyzing how volatility 𝜎 affects the price of an up-and-out call option, we find a non-linear relationship. Initially, at very low volatility (around 0.05), the option price is approximately 5, reflecting limited price fluctuations and a lower chance of hitting the barrier. As volatility increases to around 1.0, the option price peaks at approximately 30. This increase in price corresponds to greater price movements, reducing the likelihood of breaching the barrier and improving the chances of a payout. However, as volatility continues to rise beyond 1.0, the option price starts to decline. This is due to the increased probability of downward price movements, which make it more likely for the asset price to breach the barrier and render the option worthless. This highlights that moderate volatility increases the option's value, while excessive volatility significantly increases the risk of knock-out.

Une image contenant Tracé, ligne, texte, diagramme

Description générée automatiquement

**Exploring the Effect of Time to Maturity on Option Price:** As the time to maturity T increases, the price of the up-and-out call option initially rises sharply from 0 to approximately 14.5 around year 3, reflecting an extended time horizon for favorable price movements without breaching the barrier. However, after this peak, the option price begins to decline gradually, approaching close to 0 by around 90 years. This decline is driven by the increasing likelihood of breaching the barrier over a longer time horizon, which reduces the option's value. The results show that beyond a certain time frame (around 3 to 5 years), the risk of barrier breach outweighs the potential for profit, emphasizing the importance of timing when trading barrier options. This pattern of a sharp rise followed by a slower decline illustrates the sensitivity of barrier option pricing to the time to maturity, with a noticeable skew where moderate maturities provide the most value before the probability of a breach dominates over longer periods.

Une image contenant texte, ligne, Tracé, diagramme

Description générée automatiquement

## Comparative Analysis: Barrier Option Pricing – Monte Carlo Simulations vs. Approximate Closed-Form Solutions

In this section, we will compare the results obtained through **Monte Carlo simulations** with those from **approximate closed-form solutions** for barrier options. This comparison helps validate the accuracy of the simulation method and highlights the advantages and limitations of each approach.

**Barrier Option Closed-Form Solutions:** While exotic options, including barrier options, can often be too complex to price with exact closed-form solutions, there are **approximate analytical solutions** available for simpler cases of barrier options, particularly under the **Black-Scholes framework**. Rubinstein and Reiner (1991) and Haug (2007) provide such solutions for standard knock-in and knock-out barrier options.

For an **up-and-out call option**, the approximate closed-form price under the Black-Scholes assumptions is given by:

***Cup and out=CBS(S0,K,T,r,σ)−CBS(S0,H,T,r,σ)***

Where:

* ***CBS(S0,K,T,r,σ)*** is the price of a standard European call option with strike K, underlying price S0​, and maturity T.
* ***CBS(S0,H,T,r,σ)*** is the price of a European call option with strike price H (the barrier), calculated as if it were a regular call option.

The price of a standard European call option, ***CBS(S0,K,T,r,σ)***,is derived from the Black-Scholes formula. The key steps in calculating this involve the following equations:

***d1= (ln(S0/K) +(r+σ^2/2)\*T)/σ√T***

***d2=d1−σ√T***

***CBS(S0,K,T,r,σ)=S0\*N(d1)−K⋅e^-rT\*N(d2)***

* Here, ***N(d1)*** and ***N(d2)*** represent the cumulative distribution function of the standard normal distribution.

**Comparison Setup:** To conduct a meaningful comparison, we will — under the following parameters: 𝑆0=100, *K*=100, *H*=600, *T*=1.0, *r*=0.05, *σ*=0.2, *N*=5000 simulations, and *M*=252 time steps— already used in the simulation part.

Simulate up-and-out call option prices using Monte Carlo simulations and calculate the closed-form price for the up-and-out call using the Black-Scholes closed-form solution for European call options. Compare the results for validation across varying barrier levels (*H*), volatility (*σ*), and time to maturity (*T*).

**Exploring the Effect of Barrier Levels on Option Price Convergence:** The plot compares Monte Carlo simulated prices and Black-Scholes prices for an up-and-out call option across various barrier levels. When the barrier is below the strike price *K* = 100, Monte Carlo prices are near zero since the option is often knocked out early. As the barrier approaches *H* ≈100, the option price rises sharply due to a higher probability of the option remaining active. Beyond barrier levels of 150, Monte Carlo prices converge toward the Black-Scholes price, which remains constant because it doesn't account for the barrier. At lower barrier levels, knock-outs are more frequent, significantly affecting Monte Carlo prices, while at higher barriers, the option behaves more like a standard European call. This demonstrates how sensitive barrier option pricing is to the barrier level and shows the effectiveness of Monte Carlo simulations in capturing these dynamics compared to the Black-Scholes formula.

Une image contenant texte, capture d’écran, Tracé, ligne

Description générée automatiquement

**Analysis of Monte Carlo vs. Black-Scholes Prices for Different Volatility Levels:** As volatility increases, Monte Carlo prices rise initially but then fall after reaching around 0.3, reflecting the increased likelihood of the option being knocked out. In contrast, Black-Scholes prices keep increasing with volatility, as it doesn't account for the barrier. Monte Carlo is more accurate for higher volatility scenarios, where the barrier is likely to be breached, while Black-Scholes works well for lower volatility cases.

Une image contenant texte, ligne, Tracé, diagramme

Description générée automatiquement

**Analysis of Monte Carlo vs. Black-Scholes Prices for Different Time to Maturity:** The Monte Carlo simulated prices steadily increase with time to maturity, closely tracking the Black-Scholes prices throughout. Both methods show a consistent rise in option price as the expiration period extends.

Une image contenant texte, Tracé, ligne, diagramme

Description générée automatiquement

## Conclusion

**Mathematical Conclusion**: Monte Carlo simulations offer a robust method for pricing complex path-dependent options like barrier options. The simulations account for the stochastic nature of the underlying asset price, and their convergence can be observed by increasing the number of simulations (N). For barrier options, as seen through sensitivity analyses on volatility and time to maturity, the Monte Carlo method excels in capturing scenarios where the option price is sensitive to breaching the barrier. The closed-form solutions provided by Black-Scholes serve as a useful benchmark but often miss critical aspects of the barrier, particularly under higher volatility and long time-to-maturity conditions.

Mathematically, the Monte Carlo simulations and the Black-Scholes framework align for barrier levels far above the strike price, as the knock-out condition becomes less significant. However, when the barrier approaches the strike price, Monte Carlo diverges from the Black-Scholes due to the increased probability of the barrier being breached, which is not accounted for in the closed-form solution.

**Economic Conclusion**: From an economic perspective, barrier options offer a more cost-effective alternative to traditional options by introducing conditions that reduce the likelihood of payoff, such as the knock-out condition. The sensitivity of the option price to parameters like volatility and time to maturity showcases their utility in risk management, especially in volatile markets. Investors and risk managers can strategically use barrier options to hedge against extreme market conditions while minimizing premium costs.

However, the significant drop in option value under high volatility or long maturities, as shown in the Monte Carlo simulations, suggests that barrier options can become nearly worthless when there is a high probability of the barrier being breached. Hence, understanding these sensitivities is critical for pricing and using these instruments effectively.

**Perspective**: Further research can explore the application of variance reduction techniques in Monte Carlo simulations, such as antithetic variates and control variates, to improve convergence speed and accuracy. Additionally, investigating other exotic options like double-barrier or lookback options using similar simulation approaches could provide deeper insights into path-dependent pricing. Finally, exploring machine learning algorithms to predict barrier option prices under complex market conditions may open new avenues for enhancing computational efficiency.

## References:

1. **Boyle, Phelim P. (1977)** - *Options: A Monte Carlo Approach*. This paper introduced the application of Monte Carlo simulations to option pricing, setting the groundwork for modern computational methods in financial engineering.
2. **Rubinstein, Mark, and Reiner, Eric (1991)** - *Breaking Down the Barriers*. This work provides analytical solutions for certain types of barrier options, used here for the comparison with Monte Carlo results.
3. **Haug, Espen G. (2007)** - *The Complete Guide to Option Pricing Formulas*. This book provides comprehensive insights into option pricing formulas, including for barrier options, and is a key reference for closed-form solutions.